Kepler's Laws Derived

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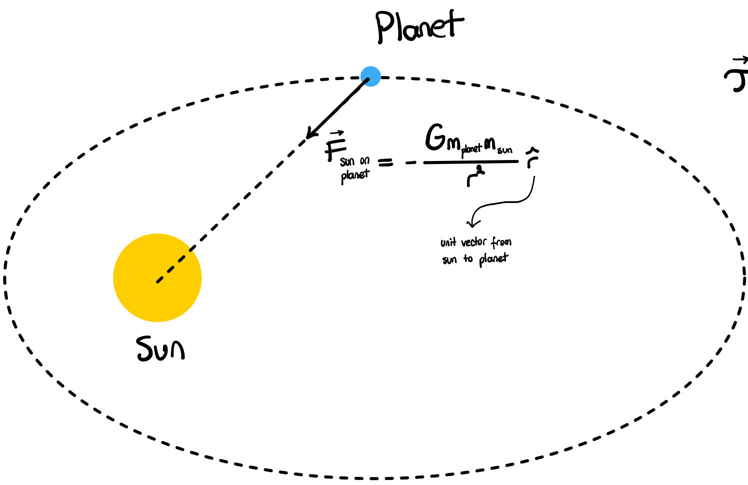
Kepler's First Law (Empiric Law)



$$\vec{\tau}_{\text{sun on planet}} = \vec{r}_{\text{from sun to planet}} \times \vec{F}_{\text{sun on planet}} = (r\hat{r}) \times \left(-\frac{Gm_{planet}m_{sun}}{r^2}\hat{r}\right) = 0$$

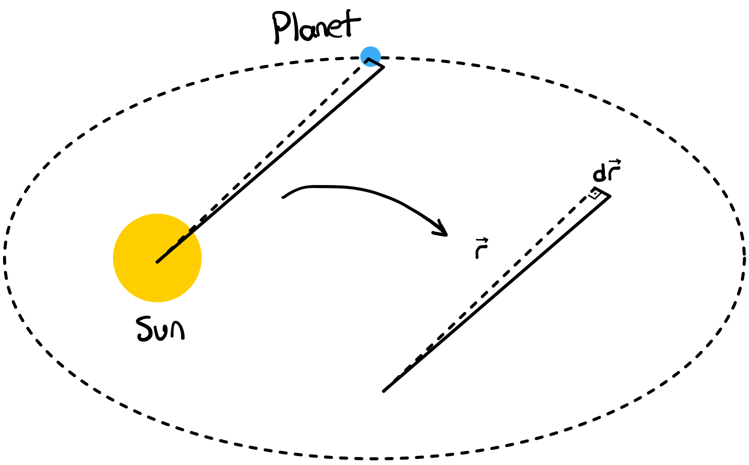
$$\vec{\tau}_{\text{sun on planet}} = \frac{d\vec{L}}{dt} = 0 \implies \vec{L} \text{ is constant}$$

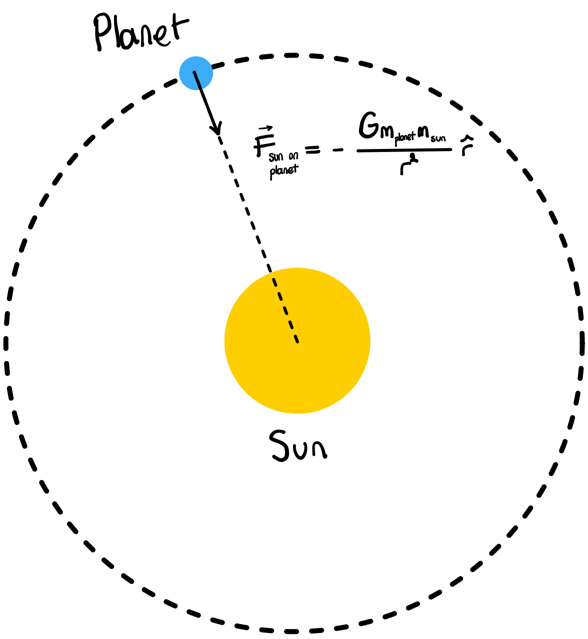
$$\vec{L} = m_{planet}\vec{r}\times\vec{v} \implies \frac{\vec{L}}{m_{planet}} = \vec{r}\times\vec{v}$$

$$dA = \tfrac{1}{2}(\vec{r}\times d\vec{r}) = \tfrac{1}{2}(\vec{r}\times d\vec{r}) = \tfrac{1}{2}(\vec{r}\times\vec{v}\,dt) = \frac{\vec{L}}{2m_{planet}}dt$$

$$\frac{dA}{dt} = \frac{\vec{L}}{2m_{planet}} \implies \underbrace{\frac{dA}{dt} \text{ is constant}}$$

Kepler's Second Law: the vector from the sun to the planet sweeps equal areas for equal time intervals.





$$\vec{F}_{\text{sun on planet}} = \vec{F}_{centripetal} = -\frac{Gm_{planet}m_{sun}}{r^2}\hat{r} = -\frac{m_{planet}v^2}{r}\hat{r} \quad \text{and} \quad v = \frac{2\pi r}{T}$$

$$\frac{Gm_{sun}}{r} = v^2 = \frac{4\pi^2 r^2}{T^2} \implies T^2 = \frac{4\pi^2 r^3}{Gm_{sun}} = k_{sun}r^3 \text{ or } T^2 \sim r^3 \text{ or } T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{Gm_{sun}}}$$

$$\underbrace{\begin{array}{cc} \underline{\text{For circular orbits}} & \underline{\text{For elliptical orbits}} \\ T^2 = k_{sun}r^3 \text{ or } T^2 \sim r^3 \text{ or } T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{Gm_{sun}}} & T^2 = k_{sun}a^3 \text{ or } T^2 \sim a^3 \text{ or } T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{Gm_{sun}}} \end{array}}$$

Kepler's Third Law

$$\left(k_{sun} \equiv \frac{4\pi^2}{Gm_{sun}} \quad a \text{ is semimajor axis} \right)$$